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Γ – *PERIODIC WAVELETS AND* $L^2(\mathbf{R}^N/\Gamma)$

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ABSTRACT. In this paper, we introduce Γ-Periodic Wavelets and give a decomposition of $L^2(R^n/\Gamma)$.

1. NOTATIONS AND SOME PRELIMINARIES

Let A $\in$ GL(n,$\mathbb{R}$) and $A^* = (A^t)^{-1}$.
Define

$$\Gamma = \{\gamma = Ak; k \in \mathbb{Z}^n\}$$

and

$$\Gamma^* = \{\gamma^* = A^*k; k \in \mathbb{Z}^n\}$$

We call Γ *the lattice with basis* A and Γ^* its dual lattice.

The set $\Omega = \Omega_\Gamma = \{x \in \mathbb{R}^n : x = At, t \in \mathbb{T}^n\}$ is called *the fundamental domain*, where $\mathbb{T} = [0,1]$.

Definition 1. A Multiresolution Analysis with lattice basis(MRALB) of $L^2(\mathbb{R}^n)$ is a family of closed subspaces, V_j(j $\in \mathbb{Z}$) of $L^2(\mathbb{R}^n)$ such that:

(1) $V_j(j \in \mathbb{Z})$ is an increasing sequence such that

$$\bigcap_{j\in\mathbb{Z}} V_j = \{0\}, \quad \overline{\bigcup_{j\in Z} V_j = L^2(\mathbb{R}^n)}$$

(2) $f(x) \in V_j$ if and only if $f(2x) \in V_{j+1}$ for all $j \in \mathbb{Z}$

(3) $f(x) \in L^2(\mathbb{R}^n)$ belongs to V_0 if and only if $f(x-\gamma) \in V_0$ for all $\gamma \in \Gamma$

(4) There exists $g \in V_0$ such that $\{g(x-\gamma); \gamma \in \Gamma\}$ is a Riesz basis of V_0. Asuume that $\{\varphi(x-\gamma) : \gamma \in \Gamma\}$ is an orthonormal basis of

V_0, then the Fourier transform of the function φ satisfies

$$\hat{\varphi}(2\xi) = m_0(\xi)\hat{\varphi}(\xi), \quad \xi \in \mathbb{R}^n$$

with $2\pi\Gamma^*$ - periodic function $m_0(\xi)$, *a filtering function.*

For $f, g \in L^2(\mathbb{R}^n)$, define

$$C(f,g)(\xi) = \sum_{\gamma\in\Gamma^*} \hat{f}(\xi + 2\pi\gamma^*)\overline{\hat{g}(\xi + 2\pi\gamma^*)}$$

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, we call it *the correlation function.* For the following Theorem 1 and Theorem 2 , see Y.Asoo[1], and [2].

Theorem 1. *Assume* $\varphi \in L^2(\mathbb{R}^n)$. *Then a system*

$$\{2^{\frac{nj}{2}}\varphi(2^j x - \gamma)\ : \gamma \in \Gamma\}$$

is an ortonormal basis of $V_j (j \in \mathbb{Z})$ *if and only if*

$$C(f,f)(\xi) \ = \ |det(A)|, \quad a.a.\ \xi \in \mathbb{R}^n.$$

Now, for a Riesz basis g of V_0 , *define the function* φ *so that*

$$\hat{\varphi}(\xi) \ = \ \sqrt{|\det(A)|}\frac{\hat{g}(\xi)}{\sqrt{C(g,g)(\xi)}},$$

then $\{2^{\frac{nj}{2}}\varphi(2^j x - \gamma)\ :\ \gamma \in \Gamma\}$ is an orthonormal basis of $V_j. Let E \ = \{0,1\}^n$ and $\psi_\epsilon \in V_1 (\epsilon \in E)$ be such that

$$\hat{\psi}_\epsilon(2\xi) \ = \ m_\epsilon(\xi)\hat{\varphi}(\xi) \tag{1.1}$$

,where $\psi_0 \ = \ \varphi$ and m_ϵ *is* $2\pi\,\Gamma^*$ - periodic.

Theorem 2. *The system* $\{\psi_\epsilon(x - \gamma)\ :\ \gamma \in \Gamma,\ \epsilon \in E\}$ *is an orthonormal basis of* V_1 *if and only if the matrix*

$$U(\xi) \ = \ (\ m_\epsilon(\xi + \pi A^* \eta)\ )_{(\epsilon,\eta)\in E^2}$$

is unitary for almost all $\xi \in \mathbb{R}^n$.

For $j \in \mathbb{Z}, \epsilon \in \tilde{E} \equiv E \setminus \{0\}$, and $\gamma \in \Gamma$,

$$\psi_{j,\epsilon,\gamma}(x) \equiv 2^{\frac{jn}{2}}\psi_\epsilon(2^j x - \gamma) \tag{1.2}$$

Define $W_{(j,\epsilon)} \ = \ \overline{\langle \psi_{(j,\epsilon,\gamma)}\ :\ \gamma \in \Gamma \rangle}$ and $W_j \ = \ \bigoplus_{\epsilon \in \tilde{E}} W_{(j,\epsilon)}$.
Then

$$L^2(\mathbb{R}^n) = V_0 \bigoplus \bigoplus_{k=0}^{\infty} W_k = \bigoplus_{k=-\infty}^{\infty} W_k \tag{1.3}$$

Definition 2. We call the system $\{\psi_{(j,\epsilon,\gamma)}; j \in \mathbb{Z}, \epsilon \in \tilde{E}, \gamma \in \Gamma\}$ wavelets basis of $L^2(\mathbb{R}^n)$, and $\{\psi_{(0,\epsilon,\gamma)}; \epsilon \in \tilde{E}, \gamma \in \Gamma\}$ mother wavelets.

In the next section , we define Γ-Periodic Wavelets and study an orthogonal decomposition of $L^2(\mathbb{R}^n/\Gamma)$.

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2. *Gamma*-PERIODIC WAVELETS AND $L^2(\mathbb{R}^n/\Gamma)$

In the following, let $\{V_j; j \in \mathbb{Z}\}$ be a MRALB with $A \in GL^+(n;\mathbb{R})$, $\{\varphi(x-\gamma); \gamma \in \Gamma\}$ be an orthonormal basis of V_0, and $\int_{\mathbb{R}^n} \varphi(x)\,dx = \sqrt{\det(A)}$.
A function $f \in L^2(\mathbb{R}^n)$ is callled $\Gamma-periodic$ if

$$f(x) = f(x+\gamma) \quad for; x \in \mathbb{R}^n, \gamma \in \Gamma.$$

Put

(1) $$P_j = P_j(\gamma) = \{f \in V_j; f \ is \ \Gamma - periodic, \ j \in \mathbb{Z}\}$$

Asuume that

(1) $P'_j s$ *are closed subspaces, and* $P_j \subset P_{j+1}$;
(2) $f(x) \in P_j$ if and only if $f(2x) \in P_{j+1}$.

Proposition 1. *For* $j \leqq 0$, $dim(P_j) = 1$, *and for* $j \geqq 1$, $dim(P_j) = 2^{nj}$.

Proof. Note that $\sum_{\gamma\in\Gamma} \varphi(x-\gamma) = \frac{1}{\sqrt{\det(A)}} \in P_0$.

For $j \leqq 0$ let $f \in P_j$ be

$$f(x) = \sum_{\gamma\in\Gamma} c(\gamma)\varphi(x-\gamma)$$

Then, for any $\gamma \in \Gamma$, $c(\gamma) = c(f) = const.$ and $f(x) = \frac{c(f)}{\sqrt{\det(A)}}$.

For $j \geqq 1$ let $f \in P_j$ then $g(x) = f(2^j x) \in P_0$.
Put $g(x) = \sum_{\gamma\in\Gamma} c(\gamma)\varphi(x-\gamma)$, then for any γ, $\gamma_0 \in \Gamma$,
$c(\gamma + 2^j\gamma_0) = c(\gamma)$, thus $dim(P_j) = 2^{nj}$. □

For $j \in \mathbb{N}$,

(2) $$\mathbb{Z}^n_{2^j} \equiv (\mathbb{Z} \mod 2^j)^n$$

(3) $$\Gamma^{(j)} \equiv \{\frac{A\,k}{2^j}; k \in \mathbb{Z}^n_{2^j}\}$$

$\Gamma^{(j)}$ is a finite additive group of order 2^{nj} and $[\Gamma^{(j+1)} : \Gamma^{(j)}] = 2^n$.

For $f \in P_j$ *and* $\gamma \in \Gamma^{(j)}$, define

$$(\gamma f)(x) = f(x-\gamma).$$

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We see that the space P_j is of dimension 2^{nj}, $\Gamma^{(j)}$ - invariant closed subspace of $L^2(\mathbb{R}^n/\Gamma) = L^2(\Omega_\Gamma)$.

For $j \in \mathbb{N}$,

$$\varphi_j(x) \equiv \sum_{\gamma\in\Gamma} 2^{\frac{nj}{2}} \varphi(2^j(x-\gamma)) \tag{4}$$

The function $\varphi_j's$ are Γ-periodic.

Theorem 1. *The system* $\{\gamma\varphi_j; \gamma \in \Gamma^{(j)}\}$, *is an orthonormal basis of the space* P_j.

Proof. Let $j = 0$, then $\varphi_0(x) = \frac{1}{\sqrt{\det(A)}}$ and

$$\int_\Omega |\varphi_0(x)|^2\, dx = 1.$$

In general, since $dim(P_j) = |\Gamma^{(j)}| = 2^{nj}$, it is sufficient to show that, for $\gamma_1,\ \gamma_2 \in \Gamma^{(j)}$,

$$\langle \gamma_1\varphi_j,\ \gamma_2\varphi_j\rangle \equiv \int_\Omega (\gamma_1\varphi_j)(x)\overline{(\gamma_2\varphi_j)(x)}\, dx = \delta(\gamma_1,\gamma_2).$$

Put $\gamma_l = \frac{A}{2^j}k_l, k_l \in \mathbb{Z}^n_{2^j},\ l = 1,2,$ then

$$\langle \gamma_1\varphi_j, \gamma_2\varphi_j\rangle = \int_\Omega \varphi_j(x - \frac{A}{2^j}k_1)\overline{\varphi_j(x - \frac{A}{2^j}k_2)}\, dx$$

$$= \sum_{l,m\in\mathbb{Z}^n} 2^{nj} \int_\Omega \varphi(2^j[x-Al] - Ak_1))\overline{\varphi(2^j[x-Am] - Ak_2)}\, dx.$$

Put $x - Al = 2^{-j}y, y \in \mathbb{R}^n$,, then we have

$$\langle \gamma_1\varphi_j, \gamma_2\varphi_j\rangle = \sum_{m\in\mathbb{R}^n} \int_{\mathbb{R}^n} \varphi(y - Ak_1)\overline{\varphi(y - A[k_2 + 2^j m])}\, dy$$

$$= \delta(k_1, k2) = \delta(\gamma_1,\gamma_2).$$

□

Let $T(\gamma)f = \gamma f, \gamma \in \Gamma^{(j)}, f \in P_j$, then $(T(\gamma), P_j)$ is a unitary representation of the group Γ^j.

Next, we consider the Fourier series expansion of the function φ_j and Poisson summation formula. Let

$$\varphi_j(x) = \sum_{l\in\mathbb{Z}^n} c(l)\exp(2\pi\imath A^* l\cdot x).$$

Then, we get

$$c(l) = \frac{1}{\det(A)}\int_\Omega \varphi_j(x)\exp(-2\pi\imath A^* l\cdot x)\, dx.$$

On the other hand,

$$\int_{\Omega} \varphi_j(x) \exp(-2\pi\imath A^* l \cdot x)\, dx$$

$$= 2^{\frac{nj}{2}} \int_{\mathbb{R}^n} \varphi(2^j x) \exp(-2\pi\imath A^* l \cdot x)\, dx$$

$$= 2^{-\frac{nj}{2}} \hat{\varphi}(2\pi \frac{A^*}{2^j} l).$$

Thus, we get

Proposition 2. *The Fourier series expansion of the function* φ_j, $j \in \mathbb{N}$ *is*

(5) $$\varphi_j(x) = \frac{1}{2^{\frac{nj}{2}} \det(A)} \sum_{\gamma^* \in \Gamma^*} \hat{\varphi}(\frac{2\pi}{2^j}\gamma^*) \exp(2\pi\imath\gamma^* \cdot x)$$

In particular, taking $x = 0$,

(6) $$2^{\frac{nj}{2}} \sum_{\gamma \in \Gamma} \varphi(2^j \gamma) = \frac{1}{2^{\frac{nj}{2}} \det(A)} \sum_{\gamma^* \in \Gamma^*} \hat{\varphi}(\frac{2\pi}{2^j}\gamma^*)$$

(Poisson Summation Formula)

Now define $\psi_{(0,0)}(x)$*as* $\frac{1}{\sqrt{\det(A)}}(= \varphi_0(x))$ *and* for $j \in \mathbb{N}$ *and* $\epsilon \in \tilde{E}$, define $\psi_{j,\epsilon}$ as

(7) $$\psi_{j,\epsilon}(x) = \frac{nj}{2} \sum_{\gamma \in \Gamma} \psi_\epsilon(2^j(x - \gamma))$$

Let $Q_j(\Gamma)$ be the orthogonal complement of $P_j(\Gamma)$ in $P_{j+1}(\Gamma)$, $dim(Q_j(\Gamma))) = 2^{nj}(2^n - 1)$.

Theorem 2. *For* $j \in \mathbb{N}$ *and* $\epsilon \in \tilde{E}$, *let* $Q_{j,\epsilon}(\Gamma)$ *be the closure of linear span of* $\{\gamma\psi_{j,\epsilon}\ ;\ \gamma \in \Gamma^{(j)}\}$.

Then, $\{\gamma\psi_{j,\epsilon}\ :\ \gamma \in \Gamma^{(j)}\}$ *is an orthonormal basis of* $Q_{j,\epsilon}(\Gamma)$ *and*

(8) $$Q_j(\Gamma) = \bigoplus_{\epsilon \in \tilde{E}} Q_{j,\epsilon}(\Gamma)$$

(9) $$P_{j+1}(\Gamma) = P_j(\Gamma) \bigoplus \bigoplus_{\epsilon \in \tilde{E}} Q_{j,\epsilon}(\Gamma)$$

(10) $$L^2(\Omega_\Gamma) = P_0(\Gamma) \bigoplus \bigoplus_{j \in \mathbb{N}} \bigoplus_{\epsilon \in \tilde{E}} Q_{j,\epsilon}(\Gamma)$$

Proof. For $\epsilon_1 \neq \epsilon_2$, $\{\gamma\psi_{\epsilon_1}(x)\ ;\ \gamma \in \Gamma\}$ and $\{\gamma\psi_{\epsilon_2}(x)\ ;\ \gamma \in \Gamma\}$ are orthogonal, so that it is sufficient to prove for $Q_{j,\epsilon}(\Gamma)$.

The rest of the proof is done in the same way to Theorem1 of this

Definition 1. We call $\{\gamma\psi_{j,\epsilon}\ ;\ \gamma \in \Gamma^{(j)}.j \in \mathbb{N}, \epsilon \in \tilde{E}\}$ a $\Gamma -$ *periodic wavelets.*

Proposition 3. *We have the Fourier series expansion*

$$\psi_{j,\epsilon}(x) = \frac{1}{2^{\frac{nj}{2}} \det(A)} \sum_{\gamma^* \in \Gamma^*} m_\epsilon(\frac{\pi\gamma^*}{2^j})\hat{\varphi}(\frac{\pi\gamma^*}{2^j}) \exp(2\pi\imath\gamma^* \cdot x) \tag{11}$$

and in particular, taking $x = 0$,

$$2^{\frac{nj}{2}} \sum_{\gamma \in Gamma} \psi_\epsilon(2^j\gamma) = \frac{1}{2^{\frac{nj}{2}} \det(A)} \sum_{\gamma^* \in \Gamma^*} m_\epsilon(\frac{\pi\gamma^*}{2^j})\hat{\varphi}(\frac{\pi\gamma^*}{2^j}) \tag{12}$$

(Poisson Summation Formula)

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